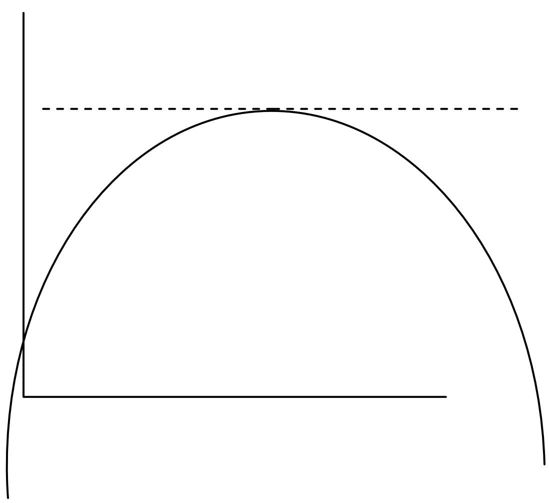
Optimisation with one variable

We are frequently interested in maximising or minimising a quantity, e.g. maximising profits or utility, or minimising costs. This can be done using differentiation.

A function is at its maximum or minimum value when it stops rising and starts falling, or vice versa. .

When a function moves from rising to falling (or v.v.), there will be a momentary **stationary point** where it is not changing.

That is, at a **local maximum** or **local minimum** of a function, the differential, F'(X), will be equal to 0 (i.e. the tangent line is flat):



We say a *local* maximum or minimum, because it may not be the *global* highest or lowest point.

Stationary points can also be *points of inflexion*, where the function flattens out then continues in the same direction. .

Example

Suppose a firm faces a demand curve given by:

$Q = 20 - 3P$

Where Q is quantity and P is price. How can the firm maximise revenue?

Well, revenue is price*quantity, PQ, which is equal to

$P(20-3P) = 20P - 3P^2$. .

So, let $F(P) = 20P - 3P^2$

Then $F'(P) = 20 - 6P$.

A stationary point will come when $F'(P) = 0$, i.e. when

$20 - 6P = 0$

Therefore, $20 = 6P$, so

$P = 20/6 = 3.333$

At this value, $Q = 20 - 3P = 10$, so revenue = 10*3.333 = 33 and a third.

Classifying stationary points

How can we be sure (apart from the graph) that this is a maximum and not a minimum or a point of inflexion? We do this by looking at the *second differential* – that is, the differential of the differential – which we write $F''(X)$ (or $\frac{d^2Y}{dX^2}$.)

E.g. if $F(X) = X^3$, then $F'(X) = 3X^2$, so $F''(X) = 3*2X = 6X$.

This is the rate of change of the rate of change.

Now at a maximum, the rate of change starts positive, goes to zero, then goes negative – so the rate of change is going down, so the rate of change of the rate of change is negative. In other words

If $F''(X)<0$ at a stationary point, then the point is a local maximum.

The opposite holds at a minimum, so

If $F''(X) > 0$ at a stationary point, the point is a local minimum.

Now in the case of our company, where the revenue function was $F(P) = 20P - 3P^2$, and $F'(P) = 20 - 6P$, with a stationary point at $P=3.333$.

Now $F''(P) = -6$. This is negative at the stationary point (indeed at all values of P), and so the point is a local maximum.

If $F''(X) = 0$ at a stationary point, the point could be a maximum, minimum or point of inflexion.

Specifically: look at successive differentials ($F'''(X)$, $F^{(4)}(X)$, etc.)

- If the <u>first</u> non-zero differential at the stationary point is of odd order (e.g. 3^{rd}, 5^{th} differential), then the stationary point is a point of inflexion.
- If the first non-zero differential at the stationary point is of even order and negative, then the stationary point is a local maximum.
- If the first non-zero differential at the stationary point is of even order and positive, then the stationary point is a local minimum.

Note that conditions for a minimum (whether in functions of one or more variables) will always be a mirror image of the conditions for a maximum. This can easily be seen, since:

Minimising the function F(X) is the same as Maximising the function –F(X).

The same holds true for functions of more than one variable.

<u>Distinguishing a global maximum or minimum</u>

In general, the global maximum or minimum can occur at any of the local maxima and minima, or at a <u>corner solution</u> – the lowest or highest possible value. (For example, a company's profits may be highest when output is zero.) . It may be necessary to look at all maxima/minima and all possible corner solutions to find the best.

However, there are certain cases where we can be sure a local maximum/minimum is the global maximum/minimum:

If $F''(X) < 0$ for the full range of values a function can take, then any local maximum is the global maximum. (We say such a function is concave).

If $F''(X) > 0$ for the full range of values a function can take, then any local minimum is the global minimum. (We say such a function is convex).

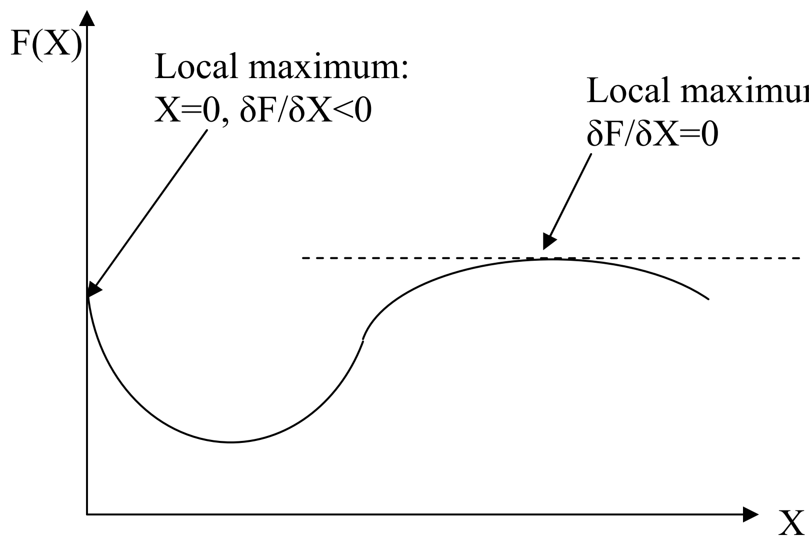
In the case we considered, we found $F''(P) = -6$, which is <0 for all possible values (0 to infinity), so the local maximum we found must be a global maximum.

Non-negativity constraints

In actual economic problems, we will frequently require that our variables should not take negative values. For example, it would not be of much use to a company to work out that its optimum number of workers is negative.

Suppose therefore that we are maximising the function $F(X)$ subject to the condition $X \geq 0$.

The (global) maximum value of $F(X)$ *either* where $\delta F/\delta X=0$ and $\delta^2 F/\delta X^2<0$, or where $X=0$ and $\delta F/\delta X \leqslant 0$. Similarily, the minimum value must occur either where $\delta F/\delta X=0$ and $\delta^2 F/\delta X^2>0$, or where $X=0$ and $\delta F/\delta X \geq 0$. We can see the reasons for this on the graph below:



A maximum at $X=0$ is known as a boundary solution, one where $X>0$ is an interior solution.

The concavity/convexity condition that guarantees that a local maximum/minimum will be a global maximum/minimum remains, for either type of local optimum.

Example: Marginal costs and marginal revenue

We know that a company maximises profits when marginal costs = marginal revenue. (MC=MR). This can be analysed in terms of calculus.

Suppose a company has a Revenue function $R(Q)$, where Q is the output, and a cost function $C(Q)$. Then the profit function, $\Pi(Q)$, can be written $\Pi(Q) = R(Q) - C(Q)$.

Differentiating, $\Pi'(Q) = R'(Q) - C'(Q)$. This will have a stationary point where $\Pi'(Q) = 0$, so $R'(Q) - C'(Q) = 0$, and hence:

$R'(Q) = C'(Q)$.

But R'(Q) is the rate of change of revenue as output increases, in other words, the marginal revenue. C'(Q) is the rate of change of costs, in other words, the marginal cost. Hence, the equation we have tells us that MC=MR.